Cospectral mates for generalized Johnson and Grassmann graphs

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Joint work with Aida Abiad, Jozefien D'haeseleer and Willem H. Haemers

Cospectral mates

Figure: Saltire pair

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$.

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Definition

Graphs with the same spectrum are **cospectral**.
Cospectral nonisomorphic graphs are **cospectral mates**.

Definition

A graph is **determined by its spectrum (DS)** if it has no cospectral mate. Otherwise, we it is **not determined by its spectrum (NDS)**.

Conjecture (Haemers)

Almost all graphs are determined by their spectrum.

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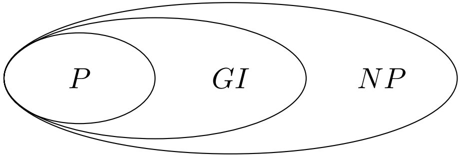

Figure: Is graph isomorphism an easy problem? Is it NP-complete?

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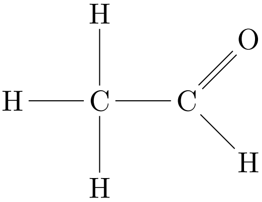
- Interesting for complexity theory
- Interesting for chemistry



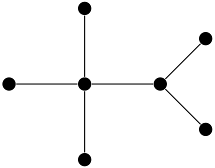


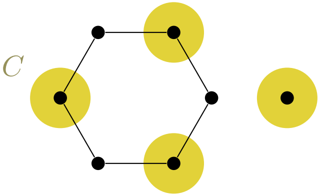
Figure: The molecular graph of acetaldehyde (ethanal).

Theorem (Godsil and McKay, 1982)

Let Γ be a graph with a subgraph C such that:

- *C is regular.*
- *Every vertex outside C has 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.*

For every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse its adjacencies with C. The resulting graph is cospectral with Γ.



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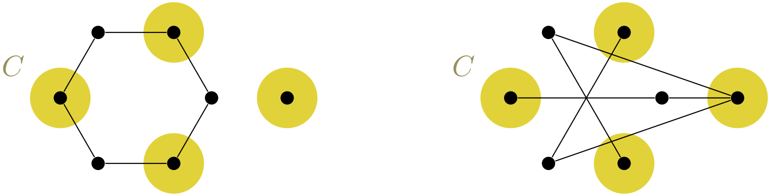

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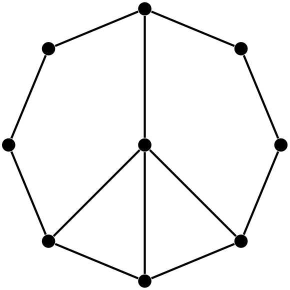
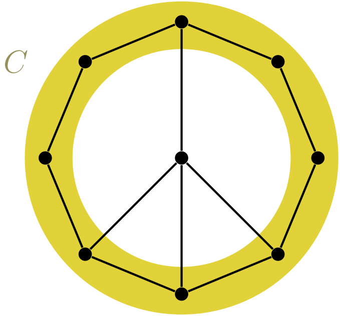
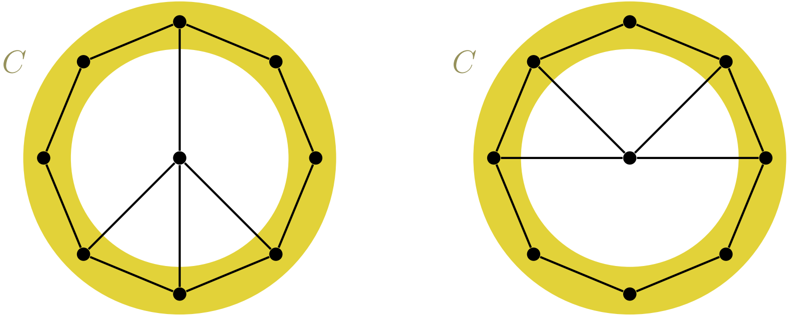
Proof.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{2}{|C|}J - I & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \frac{2}{|C|}J - I & O \\ O & I \end{pmatrix}.$$




C

How to find cospectral graphs

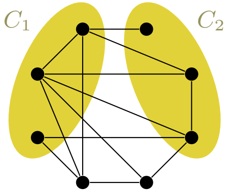


Theorem (Wang, Qiu and Hu, 2019)

Let Γ be a graph with disjoint subgraphs C_1, C_2 such that:

- $|C_1| = |C_2|$.
- *There is a constant c such that, for every vertex of C_i, the number of neighbours in C_i minus the number of neighbours in C_j, is c.*
- *Every vertex outside $C_1 \cup C_2$ has either:*
  1. 0 *neighbours in* C_1 *and* $|C_2|$ *in* C_2,
  2. $|C_1|$ *neighbours in* C_1 *and* 0 *in* C_2,
  3. *equally many neighbours in* C_1 *and* C_2.

For every $v \notin C_1 \cup C_2$ for which 1 or 2 holds, reverse its adjacencies with $C_1 \cup C_2$. The resulting graph is cospectral with Γ.



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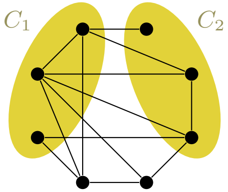

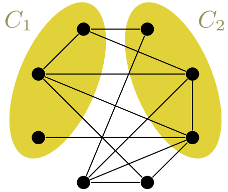

Which graphs did we check?

Definition

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Johnson graph* $J_S(n, k)$ has as vertices the k-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if their intersection size is in S.

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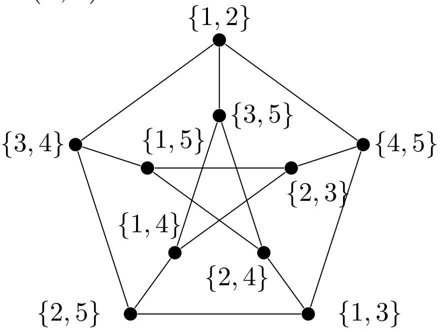
- $J_{\{0\}}(n, k)$ is the *Kneser graph* $K(n, k)$.



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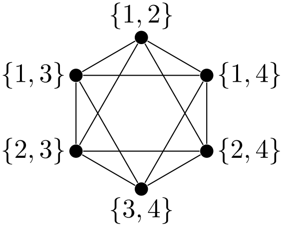
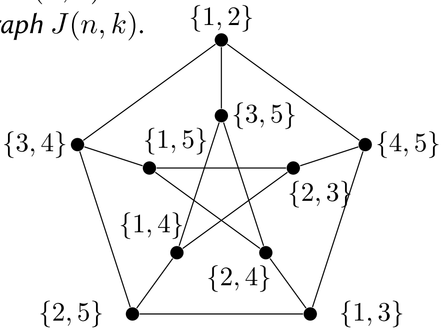
- $J_{\{0\}}(n,k)$ is the *Kneser graph* $K(n,k)$.
- $J_{\{k-1\}}(n,k)$ is the *Johnson graph* $J(n,k)$.





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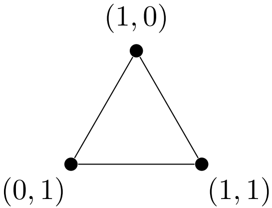
- $J_{q,\{0\}}(n,k)$ is the *q-Kneser graph* $K_q(n,k)$.



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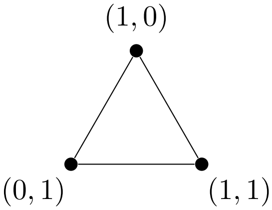
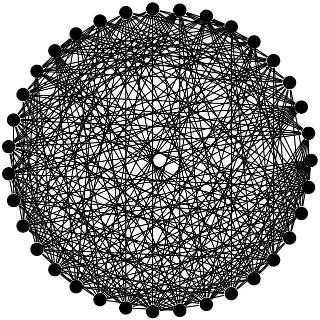
- $J_{q,\{0\}}(n,k)$ is the q-*Kneser graph* $K_q(n,k)$.
- $J_{q,\{k-1\}}(n,k)$ is the *Grassmann graph* $J_q(n,k)$.





What is known?

$J_S(n,2)$		S
		$\{0\}$
n	4	DS
	5	DS
	6	DS
	7	DS
	8	NDS
	9	DS

$J_S(n,3)$		S		
		$\{0\}$	$\{1\}$	$\{2\}$
n	6	DS	NDS	NDS
	7	DS	NDS	NDS
	8	NDS	NDS	NDS
	9	?	NDS	NDS
	10	?	NDS	NDS
	11	?	NDS	NDS

Legend: Trivial Hoffman/Chang (1959) Huang, Liu (1999) Van Dam et al. (2006) Haemers, Ramezani (2010)

What is known?

$J_S(n,4)$		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	DS	?	?	NDS	?	NDS	?
	9	DS	?	?	NDS	NDS	NDS	?
	10	?	?	?	NDS	?	NDS	?
	11	NDS	?	?	NDS	?	NDS	?
	12	?	?	?	NDS	?	NDS	?
	13	?	?	?	NDS	?	NDS	?

Legend: Trivial | Huang, Liu (1999) | Van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018)

What is known?

$J_S(n,4)$		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	DS	?	NDS	NDS	?	NDS	?
	9	DS	?	NDS	NDS	NDS	NDS	?
	10	?	?	NDS	NDS	?	NDS	?
	11	NDS	NDS	NDS	NDS	?	NDS	?
	12	?	?	NDS	NDS	?	NDS	?
	13	?	?	NDS	NDS	?	NDS	?

Legend: Trivial · Huang, Liu (1999) · Van Dam et al. (2006) · Haemers, Ramezani (2010) · Cioabă et al. (2018) · New result: $J_{\{2\}}(n,4)$ is NDS · Sporadic result

What is known?

$J_{q,S}(n,2)$		$q=2$, $S=\{0\}$	$q=3$, $S=\{0\}$	$q=4$, $S=\{0\}$
n	4	NDS	NDS	NDS
	5	NDS	NDS	NDS
	6	NDS	NDS	NDS
	7	NDS	NDS	NDS
	8	NDS	NDS	NDS
	9	NDS	NDS	NDS

Legend: Van Dam, Koolen (2005) Ihringer, Munemasa (2019)

What is known?

$J_{q,S}(n,3)$		$q=2$, S			$q=3$, S			$q=4$, S		
		$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$
n	6	?	?	NDS	?	?	NDS	?	?	NDS
	7	?	?	NDS	?	?	NDS	?	?	NDS
	8	?	?	NDS	?	?	NDS	?	?	NDS
	9	?	?	NDS	?	?	NDS	?	?	NDS
	10	?	?	NDS	?	?	NDS	?	?	NDS
	11	?	?	NDS	?	?	NDS	?	?	NDS

Legend: Van Dam et al. (2006)

What is known?

$J_{q,S}(n,3)$		$q=2$, S: $\{0\}$	$q=2$, S: $\{1\}$	$q=2$, S: $\{2\}$	$q=3$, S: $\{0\}$	$q=3$, S: $\{1\}$	$q=3$, S: $\{2\}$	$q=4$, S: $\{0\}$	$q=4$, S: $\{1\}$	$q=4$, S: $\{2\}$
n	6	NDS	?	NDS	?	?	NDS	?	?	NDS
	7	NDS	?	NDS	?	?	NDS	?	?	NDS
	8	NDS	?	NDS	?	?	NDS	?	?	NDS
	9	NDS	?	NDS	?	?	NDS	?	?	NDS
	10	NDS	?	NDS	?	?	NDS	?	?	NDS
	11	NDS	?	NDS	?	?	NDS	?	?	NDS

Legend: Van Dam et al. (2006) New result: $K_2(n,k)$ is NDS

Three new results

Theorem

$J_{\{2\}}(n,4)$ is NDS if $n \geq 8$.

Theorem

$J_{\{1,2,\dots\frac{k-1}{2}\}}(2k,k)$ is NDS if $k \geq 5$, k odd.

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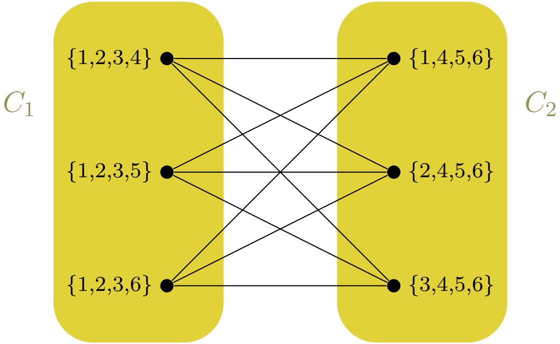
- WQH-switching



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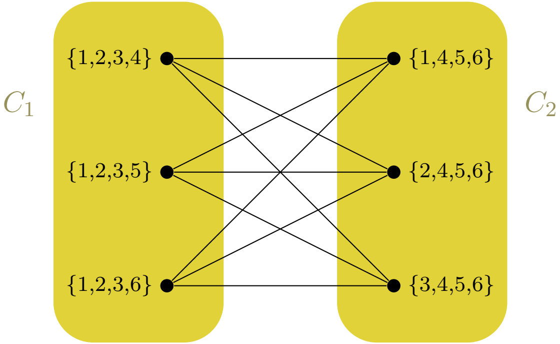

- $J_{\{2\}}(n,4)$ is edge-regular, the new graph is not

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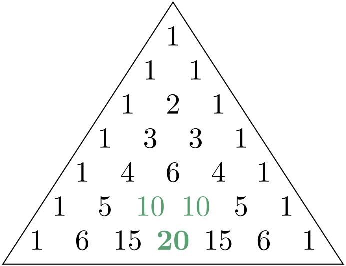
- $\binom{2k}{k} = \binom{2k-1}{k-1} + \binom{2k-1}{k} = 2\binom{2k-1}{k-1}$



Three new results

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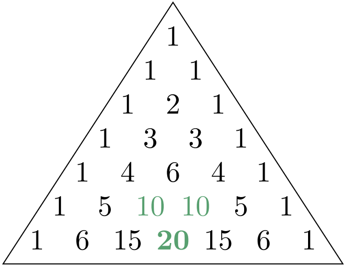
- $\binom{2k}{k} = \binom{2k-1}{k-1} + \binom{2k-1}{k} = 2\binom{2k-1}{k-1}$
- adjacency matrix $A = \begin{pmatrix} A' & \bar{A}' \\ \bar{A}' & A' \end{pmatrix}$



Theorem (Cioabă et al. (2018))

$J_{\{0,1,\dots,\frac{k-3}{2}\}}(2k-1,k-1)$ *is NDS if* $k \geq 5$, k *odd.*

$J_{\{1\}}(6,3)$ has vertices $\{1,2,3\},\{1,2,4\},\ldots,\{1,4,6\},\{1,5,6\},$
$\{4,5,6\},\{3,5,6\},\ldots,\{2,3,5\},\{2,3,4\}$

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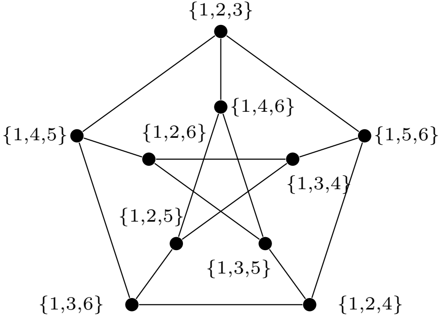

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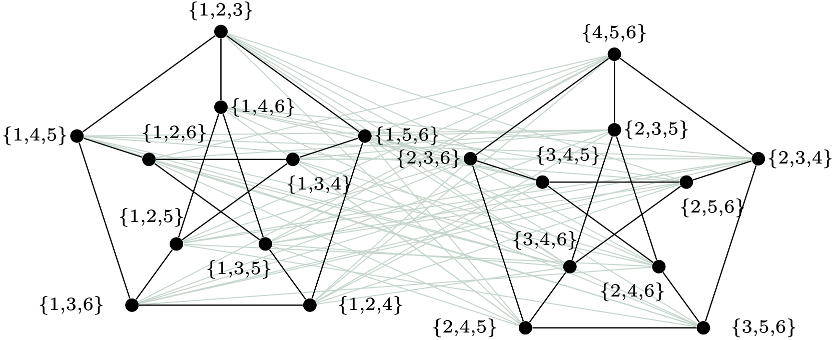

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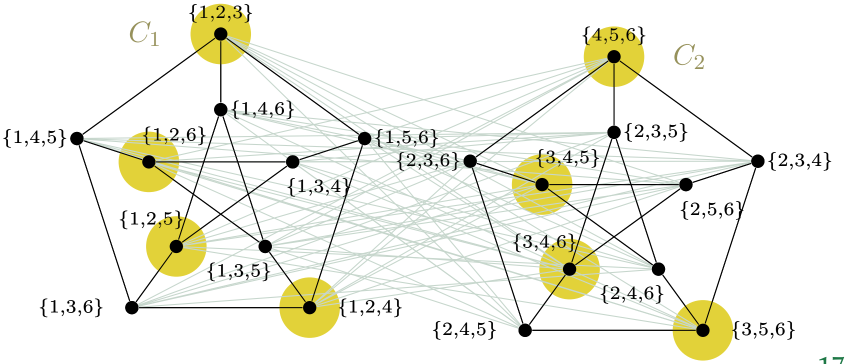

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$K_2(n,k)$ *is NDS.*

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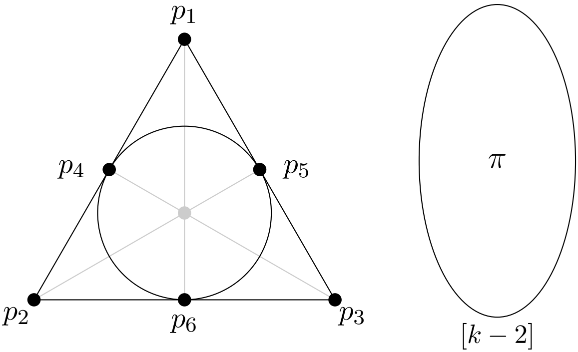

- GM-switching set $C := \{p_1p_2\pi,\ p_1p_3\pi,\ p_2p_3\pi,\ p_4p_5\pi\}$

Three new results

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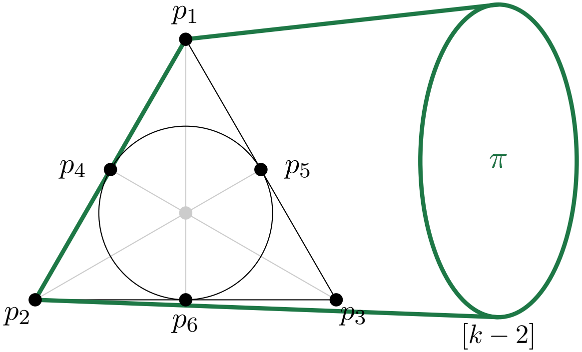

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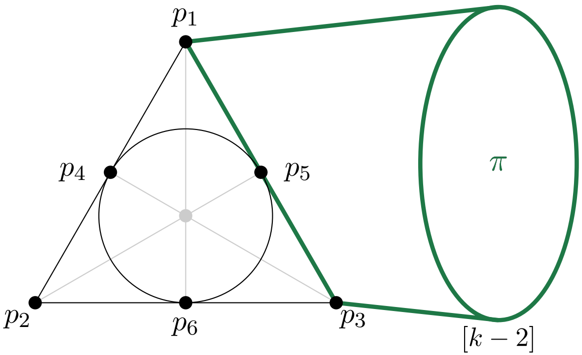

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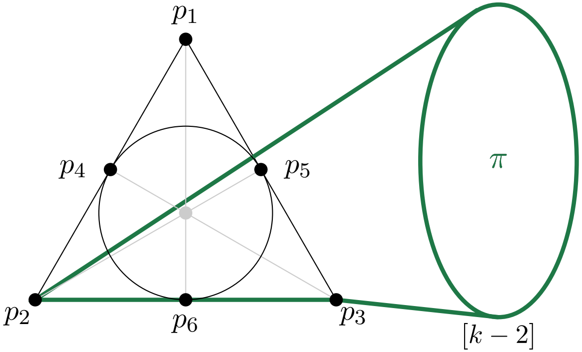

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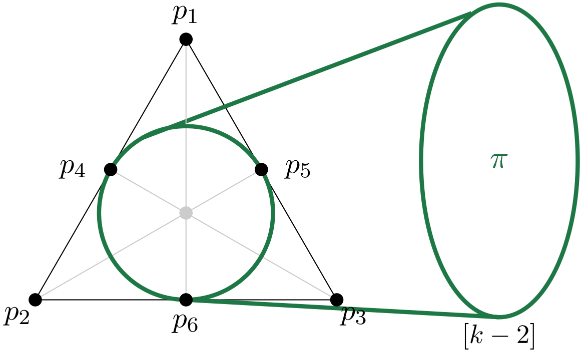

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- GM-switching, WQH-switching, AH-switching

Further research

- GM-switching, WQH-switching, AH-switching
- Computer results:

Theorem

The following graphs are NDS:

- $J_{\{1\}}(11,4)$,
- $J_{\{2,4\}}(10,5)$,
- $J_{\{2,4\}}(12,6)$.

can they be extended to infinite families?

Thank you for listening!